Quantum fluctuations and Hawking radiation around black hole horizons in Bose–Einstein condensates

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(Conformal) Schwarzschild flow — Acoustic black holes

Acoustic propagation in curved spacetime

The dynamics of a *nonrelativistic*, *perfect* and *barotropic* fluid is governed by the equations:

$$\begin{aligned} \partial_t \rho + \nabla\cdot(\rho \mathbf{v}) &= 0 && \text{(Continuity eq.)},\\ \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\tfrac{1}{\rho}\nabla p && \text{(Euler's eq.)},\\ p &= p(\rho) && \text{(Eq. of state)}.\end{aligned}$$

$$(\Longrightarrow \exists\,\psi,\ \mathbf{v} = \nabla\psi)$$

Acoustic approximation:

$$\begin{bmatrix}\psi(t,\mathbf{r})\\ \rho(t,\mathbf{r})\\ p(t,\mathbf{r})\end{bmatrix} \simeq \begin{bmatrix}\psi_0(t,\mathbf{r})\\ \rho_0(t,\mathbf{r})\\ p_0(t,\mathbf{r})\end{bmatrix} + \begin{bmatrix}\psi_1(t,\mathbf{r})\\ \rho_1(t,\mathbf{r})\\ p_1(t,\mathbf{r})\end{bmatrix}\varepsilon.$$

$$c^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}(\rho_0) \qquad x^\mu = (t,\mathbf{r})$$

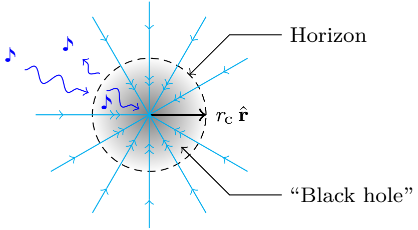
$$\begin{cases} \dfrac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu,\nu}\,\partial_\nu\right)\psi_1 = 0 \\ g_{\mu,\nu} = \dfrac{\rho_0}{c}\begin{bmatrix}(\nabla\psi_0)^2 - c^2 & -\partial_j\psi_0\\ -\partial_i\psi_0 & \delta_{i,j}\end{bmatrix}\end{cases}$$

Painlevé–Gullstrand acoustic line-element

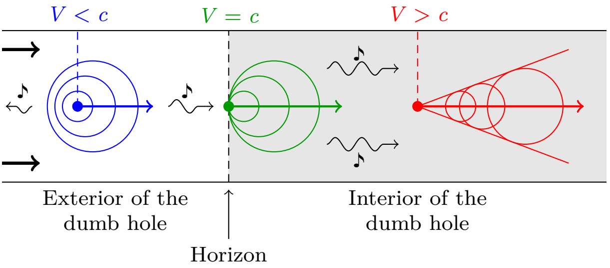
$$c = \mathrm{C}^{\mathrm{st}} \quad \text{and} \quad \nabla\psi_0(r) = -c\,\sqrt{\frac{r_{\mathrm{c}}}{r}}\,\hat{\mathbf{r}}$$

$$\mathrm{d}s^2 \propto \left(\frac{r_{\mathrm{c}}}{r}\right)^{\frac{3}{2}} \times \Big[-\left(1-\frac{r_{\mathrm{c}}}{r}\right)c^2\,\mathrm{d}t^2 + 2\sqrt{\frac{r_{\mathrm{c}}}{r}}\,c\,\mathrm{d}t\,\mathrm{d}r + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\Big]$$

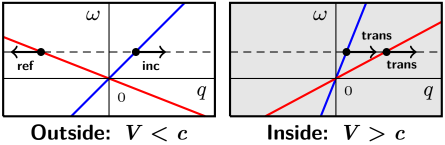
- $[\cdot] \equiv \mathrm{d}s^2_{\mathrm{Sch.}}$ (Painlevé–Gullstrand).
- $r_{\mathrm{c}} \equiv r_{\mathrm{g}} \overset{\text{def.}}{=}$ gravitational radius.
- $c \equiv c_0 \overset{\text{def.}}{=}$ vacuum speed of light.



One-dimensional dumb holes



$$\mathscr{E}_{\text{lab}}(q) = \underset{\text{(Doppler shift)}}{V\,\hbar\,q} \pm c\,\hbar\,|q| = \hbar\,\omega$$



Analog Hawking radiation

- Stimulated.— A phonon sent to the horizon gives rise to
  - two transmitted wavepackets falling down into the dumb hole and. . .
  - . . . a reflected one propagating upstream from the horizon.
- Spontaneous.— Even whithout a source, vacuum fluctuations give rise to radiation of Hawking phonons.

Dumb holes in quasi-one-dimensional Bose–Einstein condensates

Gross–Pitaevskii field equation

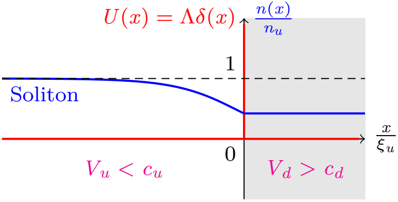
$$\mathrm{i}\hbar\,\partial_t\hat{\Psi} = -\frac{\hbar^2}{2m}\partial_{xx}\hat{\Psi} + \left[U(x) + g(x)\hat{n} - \mu\right]\hat{\Psi}$$

- $\hat{\Psi}(x,t)$: Heisenberg field operator,
- $\hat{n}(x,t) = \hat{\Psi}^\dagger\hat{\Psi}$: density operator,
- $U(x)$: external potential,
- $g(x)$: four-field coupling constant,
- μ: chemical potential.

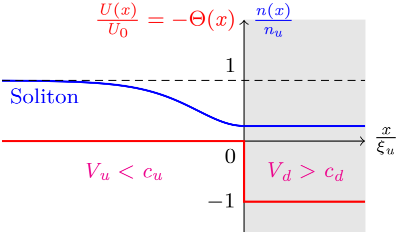
Classical stationary Gross–Pitaevskii equation

$$\mu\Psi = -\frac{\hbar^2}{2m}\partial_{xx}\Psi + \left[U(x) + g(x)n\right]\Psi$$

- $\Psi(x)$: wavefunction of the quasi-condensate [classical stationary version of $\hat{\Psi}(x,t)$],
- $n(x) = |\Psi|^2$: density [classical stationary version of $\hat{n}(x,t)$].



δ-peak configuration



Waterfall configuration

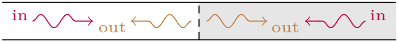
Quantum fluctuations around the background: Bogoliubov approach

$$\hat{\Psi}(x,t) = \Psi(x) + \hat{\psi}(x,t) \quad \text{with} \quad \hat{\psi} \ll \Psi$$

Bogoliubov spectrum

$$\mathscr{E}_{\text{lab}}(q) = \underbrace{V\,\hbar\,q}_{\text{(Doppler shift)}} \pm \mathscr{E}_{\text{B}}(q) = \hbar\,\omega$$

$$\mathscr{E}_{\text{B}}(q) = c\,\hbar\,|q|\,\sqrt{1 + \frac{\xi^2 q^2}{4}}$$

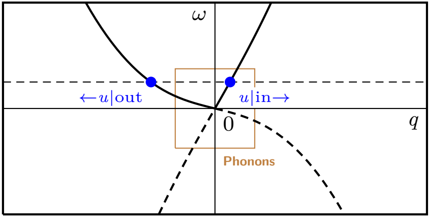


Scattering matrix

$$\begin{bmatrix} u|\text{out} \\ d1|\text{out} \\ (d2|\text{out})^\dagger \end{bmatrix} = \mathbf{S}(\omega) \begin{bmatrix} u|\text{in} \\ d1|\text{in} \\ (d2|\text{in})^\dagger \end{bmatrix}$$

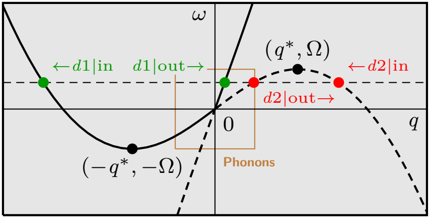
$|\mathbf{S}_{\ell,\ell'}(\omega)|^2$: transmission/reflection coefficient for a ℓ'–ingoing mode oscillating at pulsation ω scatters into a ℓ–outgoing mode.

Subsonic region



$\ell \in \{u|\text{in}, u|\text{out}, u|\text{eva}\}, \; \forall\omega > 0$

Supersonic region



$\ell \in \{d1|\text{in}, d1|\text{out}, d2|\text{in}, d2|\text{out}\}, \; \forall\omega < \Omega$

$\ell \in \{d1|\text{in}, d1|\text{out}, d|\text{eva}\}, \; \forall\omega > \Omega$

One-body Hawking signal

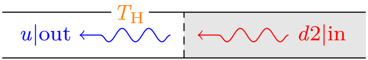
Radiated power

Energy current associated to emission of elementary excitations:

$$\hat{\Pi}(x,t) = -\frac{\hbar^2}{2m}\,\partial_t\hat{\Psi}^\dagger\,\partial_x\hat{\Psi} + \text{H.c.}.$$

Deep outside the black hole,

$$\Pi_0 \overset{\text{def.}}{=} \langle\hat{\Pi}\rangle_{T=0} = -\int_0^{\Omega}\frac{\mathrm{d}\omega}{2\pi}\,\hbar\,\omega\,|\mathbf{S}_{u,d2}(\omega)|^2.$$



Radiation spectrum

$$|\mathbf{S}_{u,d2}(\omega)|^2 \simeq \frac{\Gamma}{\exp\left(\frac{\hbar\omega}{T_\mathrm{H}}\right)-1}$$

Hawking temperature

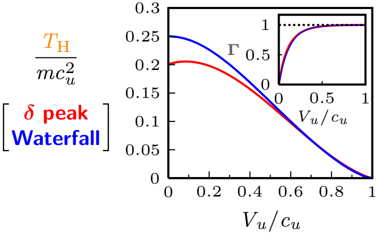
Low-ω behaviour of $\mathbf{S}_{u,d2}$:

$$\mathbf{S}_{u,d2}(\omega) \simeq f_{u,d2}\left(\frac{\hbar\omega}{mc_u^2}\right)^{-\frac{1}{2}} + h_{u,d2}\left(\frac{\hbar\omega}{mc_u^2}\right)^{\frac{1}{2}}.$$

$\Longrightarrow$ Analytical estimates of the grayness factor and the Hawking temperature:

$$\Gamma = -4\,\mathrm{Re}(f_{u,d2}^* h_{u,d2}), \qquad \frac{T_\mathrm{H}}{mc_u^2} = \frac{|f_{u,d2}|^2}{\Gamma}.$$

$T_\mathrm{H} \sim 10\,\mathrm{nK} \;<\; \mu \sim 100\,\mathrm{nK}$

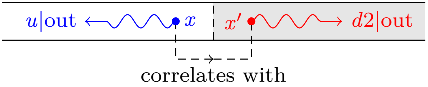


Two-body Hawking signal

Connected two-body density matrix

$g^{(2)}(x, x')$
$= \langle \hat{\Psi}^\dagger(x,t)\hat{\Psi}^\dagger(x',t)\hat{\Psi}(x,t)\hat{\Psi}(x',t)\rangle - n(x)n(x')$
$= n(x)\, n(x')\, G^{(2)}(x, x')$

Correlated phonons



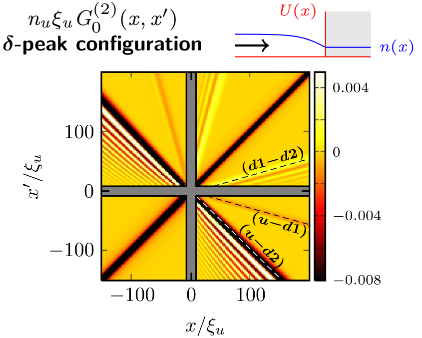
At time t after their emission the phonons $u|\text{out}$ and $d2|\text{out}$ are respectively located at

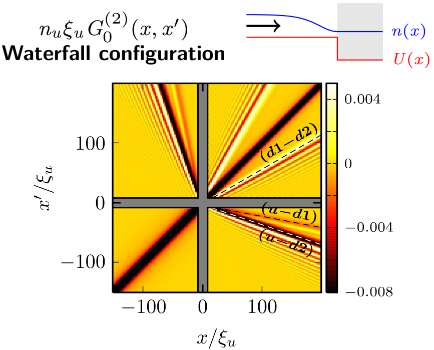
$$x = V_{\mathrm{g}}(q_{u|\text{out}})t \quad \text{and} \quad x' = V_{\mathrm{g}}(q_{d2|\text{out}})t,$$

inducing a correlation signal $(u - d2)$ along the line of slope

$$\frac{x'}{x} = \frac{V_{\mathrm{g}}(q_{d2|\text{out}})}{V_{\mathrm{g}}(q_{u|\text{out}})}$$

in the $\{x, x'\}$ plane.





Compressibility sum rule at zero temperature

In the absence of black hole

$$\int_{\mathbb{R}} \mathrm{d}x' \, g_0^{(2)}(x,x') = -n(x)$$

In the presence of black hole

The shape of the short-range anti-bunching is modified:

$$\int_{\mathbb{R}} \mathrm{d}x' \, g_0^{(2)}(x,x') \hookleftarrow -n(x) + (\text{terms})_{\text{BH}}.$$

Long-range correlations allow us to recover the sum rule:

$$\int_{\mathbb{R}} \mathrm{d}x' \, g_0^{(2)}(x,x') \hookleftarrow -(\text{terms})_{\text{BH}}.$$

Because of the sum rule,

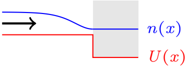
(Long-range correlations)

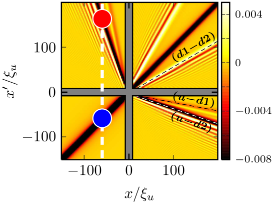
$\longleftrightarrow$

(Modifications of short-range correlations).

$n_u \xi_u \, G_0^{(2)}(x,x')$

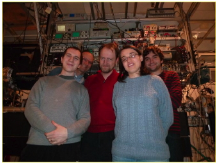
Waterfall (for example)





- $\int_{\mathbb{R}} \mathrm{d}x' \, g_0^{(2)}(x,x') \xrightarrow[x\to-\infty]{}$
  $-n_u + \frac{n_u}{2}\sqrt{\frac{c_u}{c_d}\frac{n_d}{n_u}}\,\mathrm{Re}\left(\frac{f^*_{u,d2}}{1-m_u}\mathscr{F}\right)$
- $\int_{\mathbb{R}} \mathrm{d}x' \, g_0^{(2)}(x,x') \xrightarrow[x\to+\infty]{}$
  $-n_d + \frac{n_d}{2}\left(\frac{c_u}{c_d}\right)^2 \mathrm{Re}\left[\left(\frac{f^*_{d1,d2}}{m_d+1} + \frac{f^*_{d2,d2}}{m_d-1}\right)\mathscr{F}\right]$
- $\mathscr{F} = f_{u,d2}\sqrt{\frac{c_d}{c_u}\frac{n_u}{n_d}} + f_{d1,d2} + f_{d2,d2} = \mathbf{0}$
- $m_\alpha \overset{\text{def.}}{=} V_\alpha / c_\alpha \;\; (\alpha = u, d)$

Bose–Einstein condensates of metastable helium



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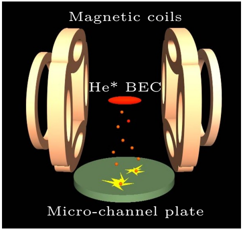
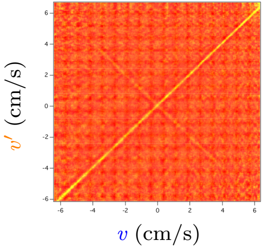




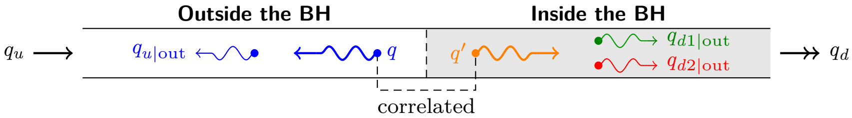
$$g^{(2)}(q, q') = \langle \hat\psi^\dagger(q,t)\,\hat\psi(q,t)\,\hat\psi^\dagger(q',t)\,\hat\psi(q',t)\rangle - \langle \hat\psi^\dagger(q,t)\,\hat\psi(q,t)\rangle\langle \hat\psi^\dagger(q',t)\,\hat\psi(q',t)\rangle$$
$$= g^{(2)}_0(q,q') + g^{(2)}_{\neq 0}(q,q')$$

$$\xi q = \frac{v}{c}$$

$$\begin{cases} g^{(2)}_0(q,q') = \frac{1}{\xi^2 q^2 (4+\xi^2 q^2)}\left(\delta_{q,q'} + \delta_{q,-q'}\right) \\ g^{(2)}_{\neq 0}(q,q') = \left[\frac{1}{4}\,\frac{(2+\xi^2 q^2)^2}{\xi^2 q^2 (4+\xi^2 q^2)}\,\delta_{q,q'} + \frac{1}{\xi^2 q^2(4+\xi^2 q^2)}\,\delta_{q,-q'}\right]\sinh^{-2}\left(\frac{g n_0}{T}\,\frac{\xi q}{2}\sqrt{1+\frac{\xi^2 q^2}{4}}\right) \end{cases}$$

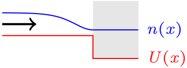
- Infrared divergences...
- Range of correlations along the line $\{q, q\}$: $\sim \xi^{-1}$?
- Width of the correlation lines $\{q, q\}$ and $\{q, -q\}$: $\sim L^{-1}, \ell_\varphi^{-1}$?

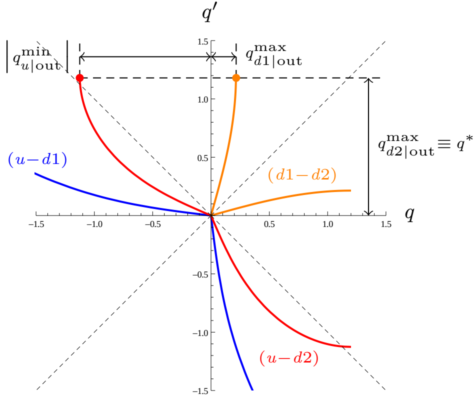
Two-body Hawking signal in momentum space

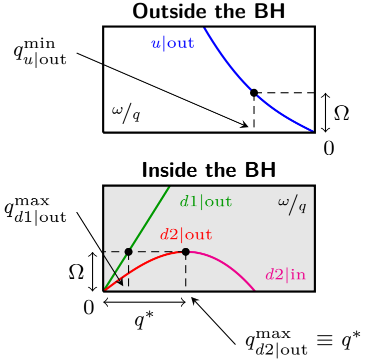


$g^{(2)}(q, q')$

Waterfall (for example)







$$\xi_d\, q^* = \left(-2 + \frac{m_d^2}{2} + \frac{m_d}{2}\sqrt{8 + m_d^2} \right)^{\frac{1}{2}}$$

Hawking radiation in quasi-1D Bose–Einstein condensates
Conclusion

• Bose–Einstein condensates offer interesting prospects to observe a *spontaneous*—so *fully quantum*—Hawking radiation.

• New dumb-hole configurations of experimental interest.

• Analytical formula for the Hawking temperature $T_{\rm H}$. $(T_{\rm H} \sim 10\,{\rm nK}) < (\mu \sim 100\,{\rm nK})$: the one-body Hawking signal is lost in the thermal noise, but…

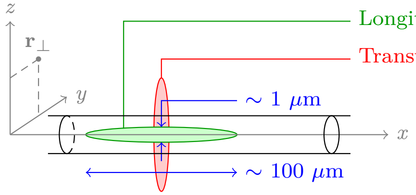
• …nonlocal density correlations (in the real space and also in the momentum space) provide a clear qualitative signature of the two-body Hawking signal, even at finite temperature.

• The usual normalization sum rule at zero temperature is also verified in the presence of an acoustic horizon: long-range density correlations have to be associated to short-range modifications of the connected two-body density matrix.

• *Prospects*: Acoustic horizon in the flow of a polariton condensate: emission of a "Hawking polarization-flux" from the horizon with $T_{\rm H}|_{\rm Polariton\ BEC} > \mu|_{\rm Polariton\ BEC}$.

P.-É. Larré, A. Recati, I. Carusotto, N. Pavloff, *PRA* (2012).
P.-É. Larré, N. Pavloff, A. M. Kamchatnov, *PRB* (2012).

Supplementary 1: Quasi-1D BECs in the mean-field regime



Harmonic radial confinement: $V_\perp(\mathbf{r}_\perp) = \frac{1}{2} m\, \omega_\perp^2\, \mathbf{r}_\perp^2$

Born–Oppenheimer approximation

$$\Psi(\mathbf{r},t) = \Psi_\parallel(x,t) \times \Psi_\perp(\mathbf{r}_\perp, \Psi_\parallel(x,t))$$

1D Gross–Pitaevskii equation

$$i\hbar\,\partial_t \Psi_\parallel = -\frac{\hbar^2}{2m}\partial_{xx}\Psi_\parallel + \left[U_{\text{ext}}(x,t) + g_{1\text{D}}\, n_{1\text{D}} - \mu\right]\Psi_\parallel$$

- $n_{1\text{D}} = |\Psi_\parallel(x,t)|^2$: longitudinal density of the condensate.
- $g_{1\text{D}} = 2\,\hbar\,\omega_\perp\, a_s \longleftrightarrow a_s > 0$: 3D s-wave scattering length.

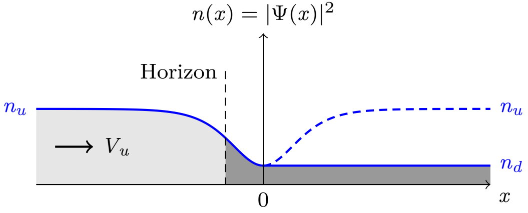
Domain of validity

$$\frac{\hbar\,\omega_\perp}{\left(\frac{\hbar^2\, a_s^{-2}}{m}\right)} \overset{(1)}{\ll} n_{1\text{D}}\, a_s \sim \frac{\mu}{\hbar\,\omega_\perp} \overset{(2)}{\ll} 1$$

- (1) allows to avoid the Tonks–Girardeau regime and implies $\mathcal{E}_{\text{int}} \ll \mathcal{E}_{\text{kin}}$ and $\ell_\varphi \gg \xi$, where $\ell_\varphi = \xi \exp \pi \sqrt{\hbar^2 n_{1\text{D}}/(g_{1\text{D}}\, m)}$.
- (2) allows to avoid the 3D-like transverse Thomas–Fermi regime and implies that the transverse motion is frozen.

Supplementary 2: Location of the acoustic horizon

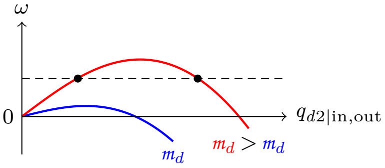
$$\Psi(x<0) = \sqrt{n_u}\left[\sqrt{1-m_u^2}\,\tanh\left(\frac{x-x_0}{\xi_u}\sqrt{1-m_u^2}\right) - \mathrm{i}m_u\right]\mathrm{e}^{\mathrm{i}q_u x}$$



Waterfall configuration $(x_0 = 0)$

$$\frac{V(x_0)}{c(x_0)} = \sqrt{\frac{2}{m_u^2} - 1} > 1$$

$$\Longrightarrow \boxed{x\big|_{\text{Horizon}} < x_0}$$

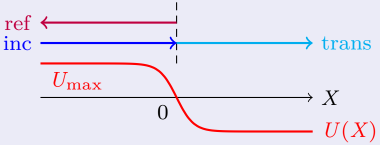


$$\Longrightarrow \boxed{x\big|_{\text{Horizon}} = x\big|_{\text{Horizon}}(\omega)}$$

Supplementary 3: Quantum reflection

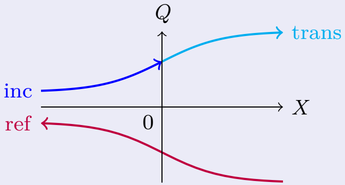
Real space

Quantum particle incoming from the left with an energy $\varepsilon > U_{\max}$.



Phase space

$$\varepsilon = \frac{Q^2}{2} + U(X) = f(X, Q)$$



— A model configuration —

$$\varepsilon = \frac{V(X)}{c} Q \pm |Q| \sqrt{1 + \frac{Q^2}{4}} = f(X, Q)$$